Medium Term Simulations of The Full Kelly and Fractional Kelly Investment Strategies

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Abstract

Using three simple investment situations, we simulate the behavior of the Kelly and fractional Kelly proportional betting strategies over medium term horizons using a large number of scenarios. We extend the work of Bicksler and Thorp (1973) and Ziemba and Hausch (1986) to more scenarios and decision periods. The results show:
(1) the great superiority of full Kelly and close to full Kelly strategies over longer horizons with very large gains a large fraction of the time;
(2) that the short term performance of Kelly and high fractional Kelly strategies is very risky;
(3) that there is a consistent tradeoff of growth versus security as a function of the bet size determined by the various strategies; and
(4) that no matter how favorable the investment opportunities are or how long the finite horizon is, a sequence of bad results can lead to poor final wealth outcomes, with a loss of most of the investor's initial capital.

1 Introduction

The Kelly optimal capital growth investment strategy has many long term positive theoretical properties (MacLean, Thorp and Ziemba 2009). It has been dubbed " fortunes formula" by Thorp (see Poundstone, 2005). However, properties that hold in the long run may be countered by negative short to medium term behavior because of the low risk aversion of log utility. In this paper, three well known experiments are revisited. The objectives are: (i) to compare the Bicksler - Thorp (1973) and Ziemba - Hausch (1986) experiments in the same setting; and (ii) to study them using an expanded range of scenarios and investment strategies. The class of investment strategies generated by varying the fraction of investment capital allocated to the Kelly portfolio are applied to simulated returns from the experimental models, and the distribution of accumulated capital is described. The conclusions from the expanded experiments are compared to the original results.

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2 Fractional Kelly Strategies: The Ziemba and Hausch (1986) example

We begin with an investment situation with five possible independent investments where one wagers \$1 and either loses it with probability $1-p$ or wins $\$(O+1)$ with probability p, where O is the odds. The five wagers with odds of $O = 1, 2, 3, 4$ *and* 5 to one all have expected value of 1.14. The optimal Kelly wagers are the expected value edge of 14% over the odds. So the wagers run from 14%, down to 2.8% of initial and current wealth at each decision point. Table 1 describes these investments. The value 1.14 was chosen as it is the recommended cutoff for profitable place and show racing bets using the system described in Ziemba and Hausch (1986).

Win Probability	Odds	Prob of Selection in Simulation	Kelly Bets
0.570	1-1	0.1	0.140
0.380	2-1	0.3	0.070
0.285	3-1	0.3	0.047
0.228	4-1	0.2	0.035
0.190	5-1	0.1	0.028

Table 1: The Investment Opportunities

Ziemba-Hausch (1986) used 700 decision points and 1000 scenarios and compared full with half Kelly strategies. We use the same 700 decision points and 2000 scenarios and calculate more attributes of the various strategies. We use full, 3/4, 1/2, 1/4, and 1/8 Kelly strategies and compute the maximum, mean, minimum, standard deviation, skewness, excess kurtosis and the number out of the 2000 scenarios that the final wealth starting from an initial wealth of \$1000 is more than \$50, \$100, \$500 (lose less than half), \$1000 (breakeven), \$10,000 (more than 10-fold), \$100,000 (more than 100-fold), and \$1 million (more than a thousand-fold). Table 2 shows these results and illustrates the conclusions stated in the abstract. The final wealth levels are much higher on average, the higher the Kelly fraction. With 1/8 Kelly, the average final wealth is \$2072, starting with \$1000. Its \$4339 with 1/4 Kelly, \$19,005 with half Kelly, \$70,991 with 3/4 Kelly and \$524,195 with full Kelly. So as you approach full Kelly, the typical final wealth escalates dramatically. This is shown also in the maximum wealth levels which for full Kelly is \$318,854,673 versus \$6330 for 1/8 Kelly.

			Kelly Fraction		
Statistic	1.0k	0.75k	0.50k	0.25k	0.125k
Max	318854673	4370619	1117424	27067	6330
Mean	524195	70991	19005	4339	2072
Min	4	56	111	513	587
St. Dev.	8033178	242313	41289	2951	650
Skewness	35	11	13	2	1
Kurtosis	1299	155	278	9	2
$> 5 \times 10$	1981	2000	2000	2000	2000
10^2	1965	1996	2000	2000	2000
$> 5 \times 10^2$	1854	1936	1985	2000	2000
$> 10^3$	1752	1855	1930	1957	1978
$> 10^4$	1175	1185	912	104	0
$> 10^5$	479	284	50	0	0
$> 10^6$	111	17	1	0	0

Table 2: Final Wealth Statistics by Kelly Fraction: Ziemba-Hausch (1986) Model

Figure 1 shows the wealth paths of these maximum final wealth levels. Most of the gain is in the last 100 of the 700 decision points. Even with these maximum graphs, there is much volatility in the final wealth with the amount of volatility generally higher with higher Kelly fractions. Indeed with 3/4 Kelly, there were losses from about decision point 610 to 670.

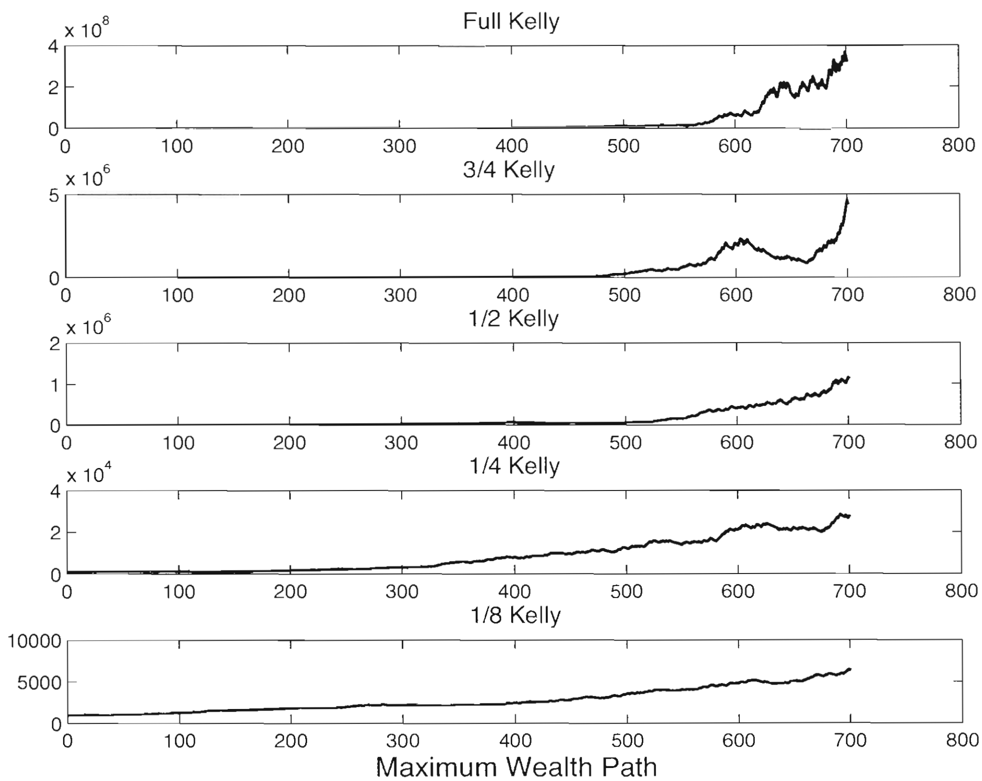


Figure 1: Highest Final Wealth Trajectory: Ziemba-Hausch (1986) Model

Looking at the chance of losses (final wealth is less than the initial $1000) in all cases, even with

1/8 Kelly with 1.1% and 1/4 Kelly with 2.15%, there are losses even with 700 independent bets each with an edge of 14%. For full Kelly, it is fully 12.4% losses, and it is 7.25% with 3/4 Kelly and 3.5% with half Kelly. These are just the percent of losses. But the size of the losses can be large as shown in the >50, >100, and >500 and columns of Table 2. The minimum final wealth levels were 587 for 1/8 and 513 for 1/4 Kelly so you never lose more than half your initial wealth with these lower risk betting strategies. But with 1/2, 3/4 and full Kelly, the minimums were 111, 56, and only $4. Figure 2 shows these minimum wealth paths. With full Kelly, and by inference 1/8, 1/4, 1/2, and 3/4 Kelly, the investor can actually never go fully bankrupt because of the proportional nature of Kelly betting.

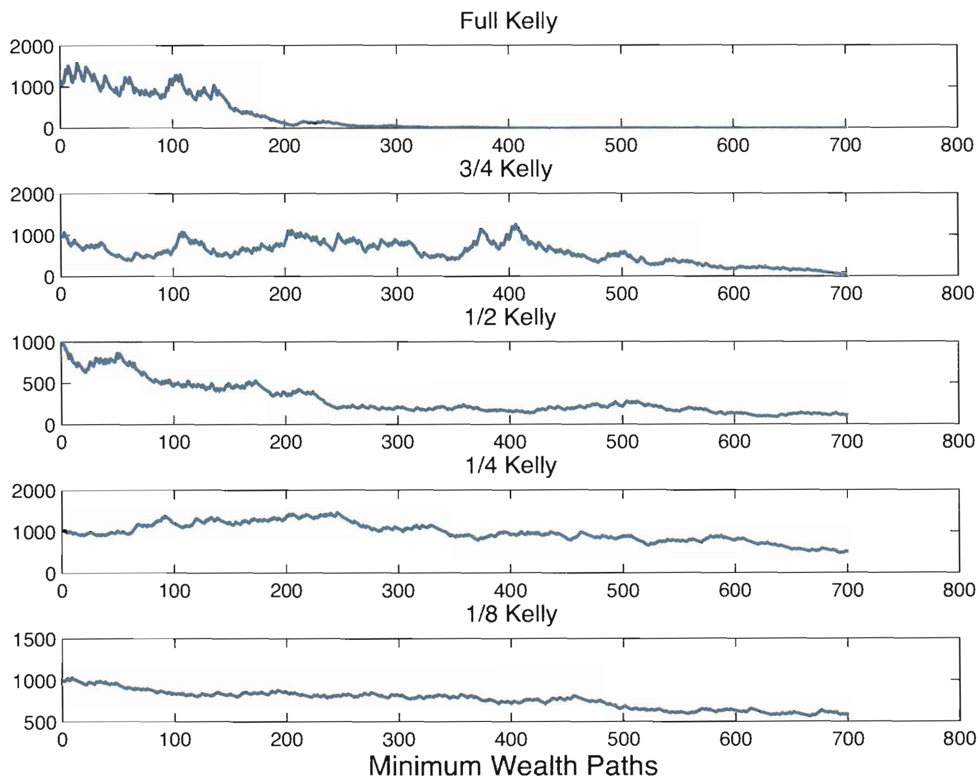


Figure 2: Lowest Final Wealth Trajectory: Ziemba-Hausch (1986) Model

If capital is infinitely divisible and there is no leveraging than the Kelly bettor cannot go bankrupt since one never bets everything (unless the probability of losing anything at all is zero and the probability of winning is positive). If capital is discrete, then presumably Kelly bets are rounded down to avoid overbetting, in which case, at least one unit is never bet. Hence, the worst case with Kelly is to be reduced to one unit, at which point betting stops. Since fractional Kelly bets less, the result follows for all such strategies. For levered wagers, that is, betting more than one's wealth with borrowed money, the investor can lose more than their initial wealth and become bankrupt.

3 Proportional Investment Strategies: Alternative Experiments

The growth and risk characteristics for proportional investment strategies such as the Kelly depend upon the returns on risky investments. In this section we consider some alternative investment experiments where the distributions on returns are quite different. The mean return is similar: 14% for Ziemba-Hausch, 12.5%for Bicksler-Thorp I, and 10.2% for Bicksler-Thorp II. However, the variation around the mean is not similar and this produces much different Kelly strategies and corresponding wealth trajectories for scenarios.

3.1 The Ziemba and Hausch (1986) Model

The first experiment is a repeat of the Ziemba - Hausch model in Section 2. A simulation was performed of 3000 scenarios over $T = 40$ decision points with the five types of independent investments for various investment strategies. The Kelly fractions and the proportion of wealth invested are reported in Table 3. Here, $1.0k$ is full Kelly, the strategy which maximizes the expected logarithm of wealth. Values below 1.0 are fractional Kelly and coincide in this setting with the decision from using a negative power utility function. Values above 1.0 coincide with those from some positive power utility function. This is overbetting according to MacLean, Ziemba and Blazenko (1992), because long run growth rate falls and security (measured by the chance of reaching a specific positive goal before falling to a negative growth level) also falls.

				Kelly Fraction: f			
Opportunity	*1.75k*	*1.5k*	*1.25k*	*1.0k*	*0.75k*	*0.50k*	*0.25k*
A	0.245	0.210	0.175	0.140	0.105	0.070	0.035
B	0.1225	0.105	0.0875	0.070	0.0525	0.035	0.0175
C	0.08225	0.0705	0.05875	0.047	0.03525	0.0235	0.01175
D	0.06125	0.0525	0.04375	0.035	0.02625	0.0175	0.00875
E	0.049	0.042	0.035	0.028	0.021	0.014	0.007

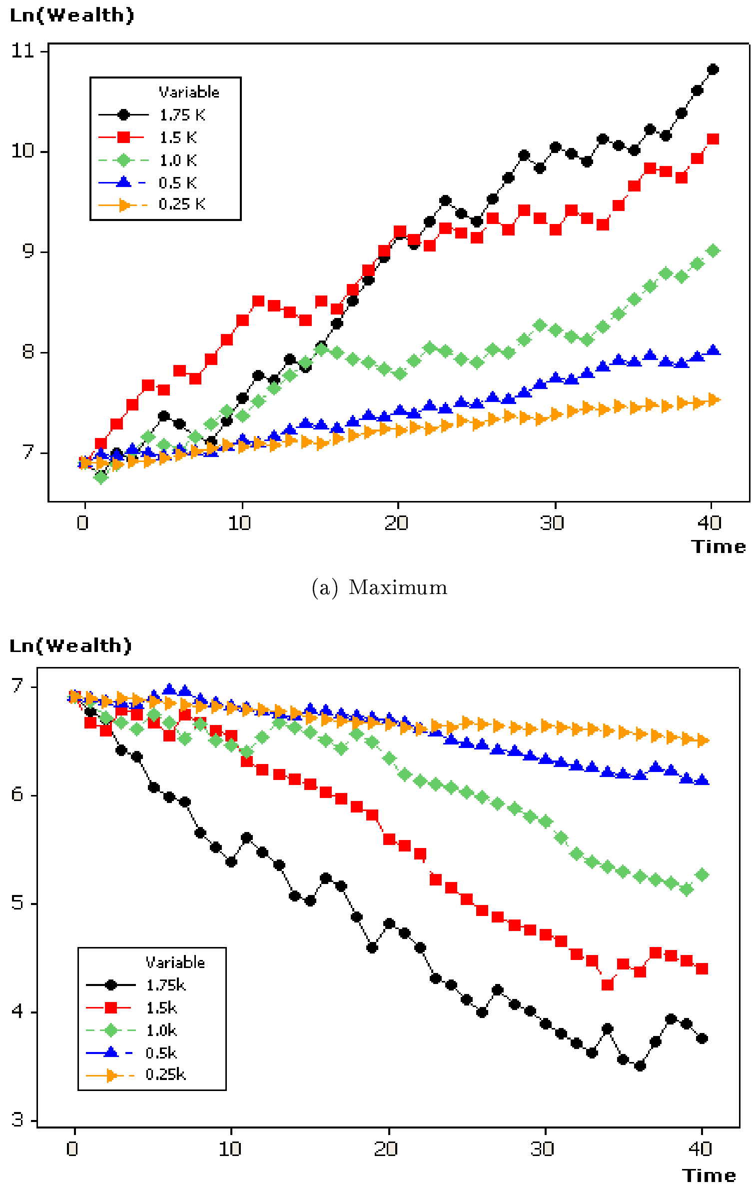
Table 3: The Investment Proportions (λ) and Kelly Fractions

The initial wealth for investment was 1000. Table 4 reports statistics on the final wealth for $T = 40$ with the various strategies.

				Fraction			
Statistic	1.75k	1.5k	1.25k	1.0k	0.75k	0.50k	0.25k
Max	50364.73	25093.12	21730.90	8256.97	6632.08	3044.34	1854.53
Mean	1738.11	1625.63	1527.20	1386.80	1279.32	1172.74	1085.07
Min	42.77	80.79	83.55	193.07	281.25	456.29	664.31
St. Dev.	2360.73	1851.10	1296.72	849.73	587.16	359.94	160.76
Skewness	6.42	4.72	3.49	1.94	1.61	1.12	0.49
Kurtosis	85.30	38.22	27.94	6.66	5.17	2.17	0.47
$> 5 \times 10$	2998	3000	3000	3000	3000	3000	3000
10^2	2980	2995	2998	3000	3000	3000	3000
$> 5 \times 10^2$	2338	2454	2634	2815	2939	2994	3000
$> 10^3$	1556	11606	1762	1836	1899	1938	2055
$> 10^4$	43	24	4	0	0	0	0
$> 10^5$	0	0	0	0	0	0	0
$> 10^6$	0	0	0	0	0	0	0

Table 4: Wealth Statistics by Kelly Fraction: Ziemba-Hausch Model (1986)

Since the Kelly bets are small, the proportion of current wealth invested is not high for any of the fractions. The upside and down side are not dramatic in this example, although there is a substantial gap between the maximum and minimum wealth with the highest fraction. Figure 3 shows the trajectories which have the highest and lowest final wealth for a selection of fractions. The log-wealth is displayed to show the rate of growth at each decision point. The lowest trajectories are almost a reflection of the highest ones.

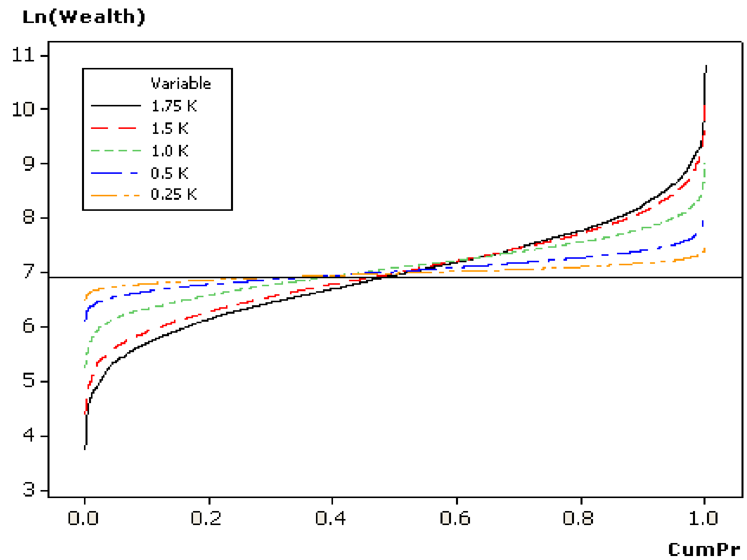


(a) Maximum

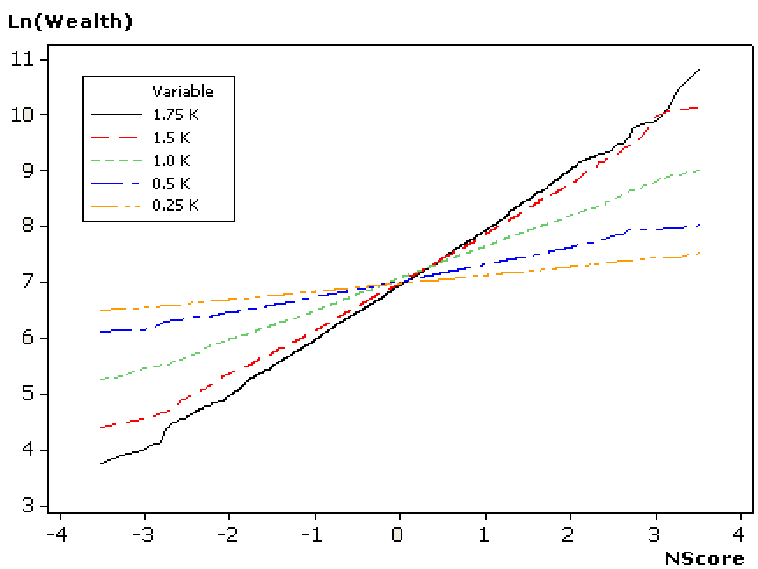
(b) Minimum

Figure 3: Trajectories with Final Wealth Extremes: Ziemba-Hausch Model (1986)

The skewness and kurtosis indicate that final wealth is not normally distributed. This is expected since the geometric growth process suggests a log-normal wealth. Figure 4 displays the simulated log-wealth for selected fractions at the horizon $T = 40$. The normal probability plot will be linear if terminal wealth is distributed log-normally. The slope of the plot captures the shape of the log-wealth distribution. In this case the final wealth distribution is close to log-normal. As the Kelly fraction increases the slope increases, showing the longer right tail but also the increase in downside risk in the wealth distribution.



(a) Inverse Cumulative



(b) Normal Plot

Figure 4: Final Ln(Wealth) Distributions by Fraction: Ziemba-Hausch Model (1986)

On the inverse cumulative distribution plot, the initial wealth $\ln(1000) = 6.91$ is indicated to show the chance of losses. The inverse cumulative distribution of log-wealth is the basis of comparisons of accumulated wealth at the horizon. In particular, if the plots intersect then first order stochastic dominance by a wealth distribution does not exist (Hanoch and Levy, 1969). The mean and standard deviation of log-wealth are considered in Figure 5, where the trade-off as the Kelly fraction varies can be understood. Observe that the mean log-wealth peaks at the full Kelly strategy whereas the standard deviation is monotone increasing. Fractional strategies greater than full Kelly are inefficient in log-wealth, since the growth rate decreases and the the standard deviation of log-wealth increases.

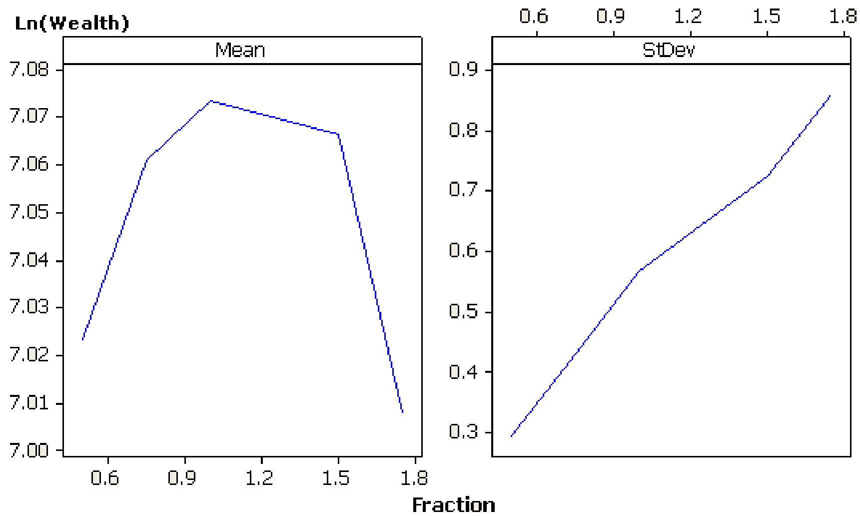


Figure 5: Mean-Std Tradeoff: Ziemba-Hausch Model (1986)

The results in Table 4 and Figures 3 - 5 support the following conclusions for Experiment 1.

1. The statistics describing end of horizon ($T = 40$) wealth are all monotone in the fraction of wealth invested in the Kelly portfolio. Specifically the maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction. In contrast the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. There is a trade-off between wealth growth and risk. The cumulative distribution in Figure 4 supports the theory for fractional strategies, as there is no dominance, and the distribution plots all intersect.

2. The maximum and minimum final wealth trajectories clearly show the wealth growth - risk trade-off of the strategies. The worst scenario is the same for all Kelly fractions so that the wealth decay is greater with higher fractions. The best scenario differs for the low fraction strategies, but the growth path is almost monotone in the fraction. The mean-standard deviation trade-off demonstrates the inefficiency of levered strategies (greater than full Kelly).

3.2 Bicksler - Thorp (1973) Case I - Uniform Returns

There is one risky asset R having mean return of $+12.5\%$, with the return uniformly distributed between 0.75 and 1.50 for each dollar invested. Assume we can lend or borrow capital at a risk free rate $r = 0.0$. Let $\lambda =$ the proportion of capital invested in the risky asset, where λ ranges from 0.4 to 2.4 . So $\lambda = 2.4$ means \$1.4 is borrowed for each \$1 of current wealth. The Kelly optimal growth investment in the risky asset for $r = 0.0$ is $x = 2.8655$. The Kelly fractions for the different values of λ are shown in Table 3. (The formula relating λ and f for this expiriment is in the Appendix.) In their simulation, Bicksler and Thorp use 10 and 20 yearly decision periods, and 50 simulated scenarios. We use 40 yearly decision periods, with 3000 scenarios.

Proportion: λ	0.4	0.8	1.2	1.6	2.0	2.4
Fraction: f	0.140	0.279	0.419	0.558	0.698	0.838

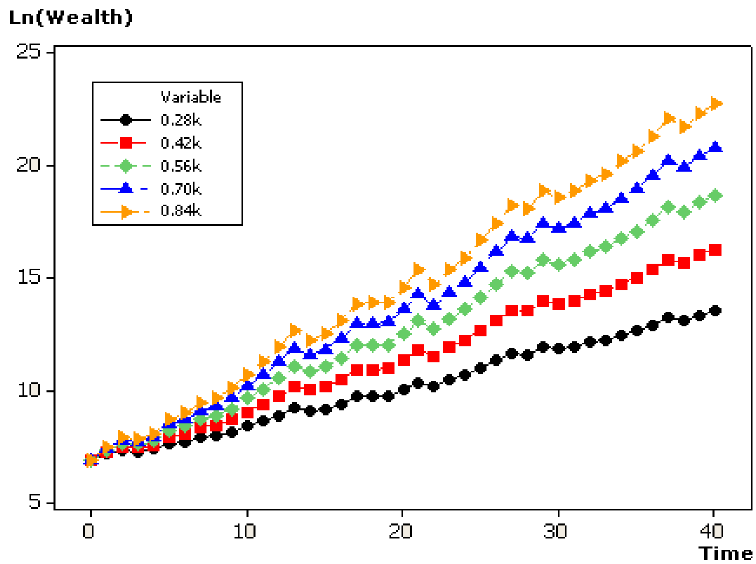
Table 5: The Investment Proportions and Kelly Fractions for Bicksler-Thorp (1973) Case I

The numerical results from the simulation with $T = 40$ are in Table 6 and Figures 7 - 9. Although the Kelly investment is levered, the fractions in this case are less than 1.

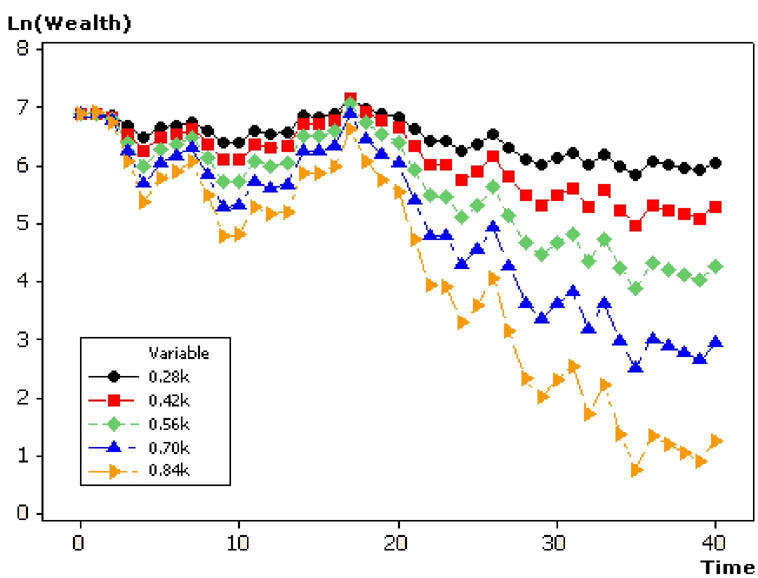
			Fraction			
Statistic	0.14k	0.28k	0.42k	0.56k	0.70k	0.84k
Max	34435.74	743361.14	11155417.33	124068469.50	1070576212.0	7399787898
Mean	7045.27	45675.75	275262.93	1538429.88	7877534.72	36387516.18
Min	728.45	425.57	197.43	70.97	18.91	3.46
St. Dev.	4016.18	60890.61	674415.54	6047844.60	44547205.57	272356844.8
Skewness	1.90	4.57	7.78	10.80	13.39	15.63
Kurtosis	6.00	31.54	83.19	150.51	223.70	301.38
$> 5 \times 10$	3000	3000	3000	3000	2999	2998
10^2	3000	3000	3000	2999	2999	2998
$> 5 \times 10^2$	3000	2999	2999	2997	2991	2976
$> 10^3$	2998	2997	2995	2991	2980	2965
$> 10^4$	529	2524	2808	2851	2847	2803
$> 10^5$	0	293	1414	2025	2243	2290
$> 10^6$	0	0	161	696	1165	1407

Table 6: Final Wealth Statistics by Kelly Fraction for Bicksler-Thorp Case I

In this experiment the Kelly proportion is high, based on the attractiveness of the investment in stock. The largest fraction (0.838k) shows strong returns, although in the worst scenario most of the wealth is lost. The trajectories for the highest and lowest terminal wealth scenarios are displayed in Figures 6. The highest rate of growth is for the highest fraction, and correspondingly it has the largest wealth fallback.



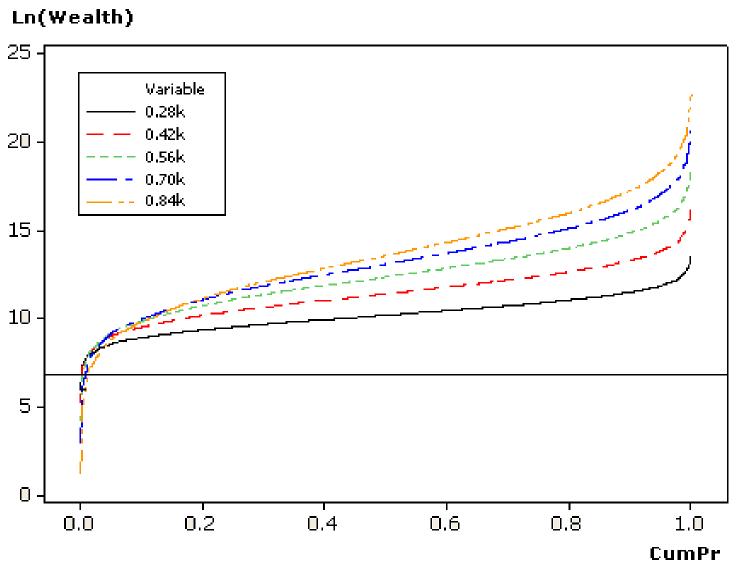
(a) Maximum



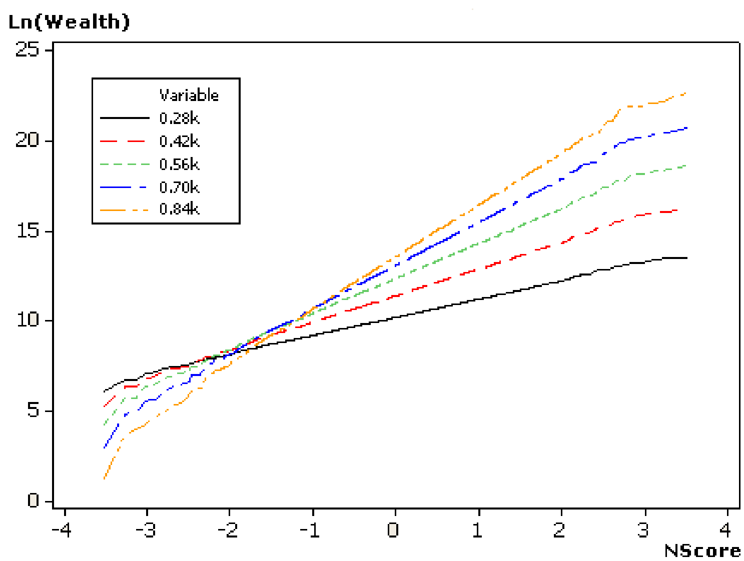
(b) Minimum

Figure 6: Trajectories with Final Wealth Extremes: Bicksler-Thorp (1973) Case I

The distribution of terminal wealth in Figure 7 illustrates the growth of the $f = 0.838k$ strategy. It intersects the normal probability plot for other strategies very early and increases its advantage. The linearity of the plots for all strategies is evidence of the log-normality of final wealth. The inverse cumulative distribution plot indicates that the chance of losses is small - the horizontal line indicates log of initial wealth.



(a) Inverse Cumulative



(b) Normal Plot

Figure 7: Final Ln(Wealth) Distributions: Bicksler-Thorp (1973) Case I

As further evidence of the superiority of the $f = 0.838k$ strategy consider the mean and standard deviation of log-wealth in Figure 8. The growth rate (mean ln(Wealth)) continues to increase since the fractional strategies are less then full Kelly.

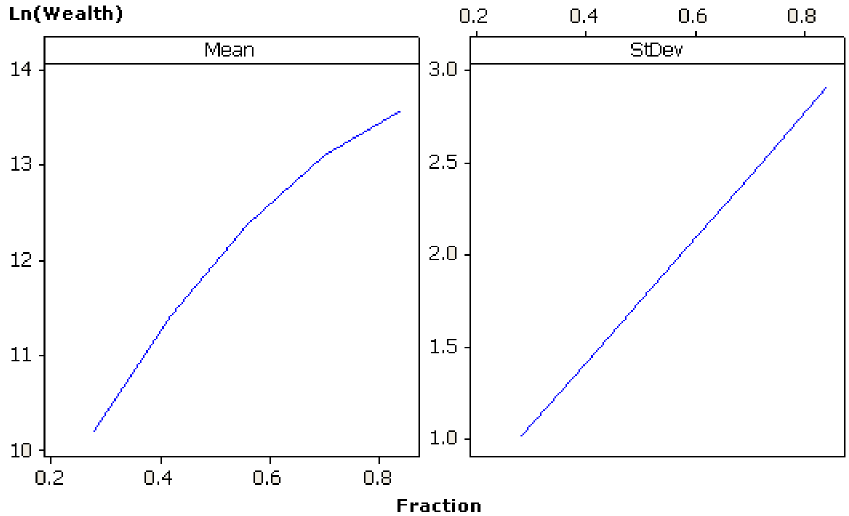


Figure 8: Mean-Std Trade-off: Bicksler-Thorp (1973) Case I

From the results of this experiment we can make the following statements.

1. The statistics describing end of horizon ($T = 40$) wealth are again monotone in the fraction of wealth invested in the Kelly portfolio. Specifically the maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction. In contrast the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. The growth and decay are much more pronounced than was the case in experiment 1. The minimum still remains above 0 since the fraction of Kelly is less than 1. There is a trade-off between wealth growth and risk, but the advantage of leveraged investment is clear. As illustrated with the cumulative distributions in Figure 7, the log-normality holds and the upside growth is more pronounced than the downside loss. Of course, the fractions are less than 1 so improved growth is expected.

2. The maximum and minimum final wealth trajectories clearly show the wealth growth - risk of various strategies. The mean-standard deviation trade-off favors the largest fraction, even though it is highly levered.

3.3 Bicksler - Thorp (1973) Case II - Equity Market Returns

In the third experiment there are two assets: US equities and US T-bills. According to Siegel (2002), during 1926-2001 US equities returned of 10.2% with a yearly standard deviation of 20.3%, and the mean return was 3.9% for short term government T-bills with zero standard deviation. We assume the choice is between these two assets in each period. The Kelly strategy is to invest a proportion of wealth $x = 1.5288$ in equities and sell short the T-bill at $1 - x = -0.5228$ of current wealth. With the short selling and levered strategies, there is a chance of substantial losses. For the simulations, the proportion: λ of wealth invested in equities and the corresponding Kelly fraction f are provided in Table 7. (The formula relating λ and f for this expiriment is in the Appendix.)

In their simulation, Bicksler and Thorp used 10 and 20 yearly decision periods, and 50 simulated scenarios. We use 40 yearly decision periods, with 3000 scenarios.

λ	0.4	0.8	1.2	1.6	2.0	2.4
f	0.26	0.52	0.78	1.05	1.31	1.57

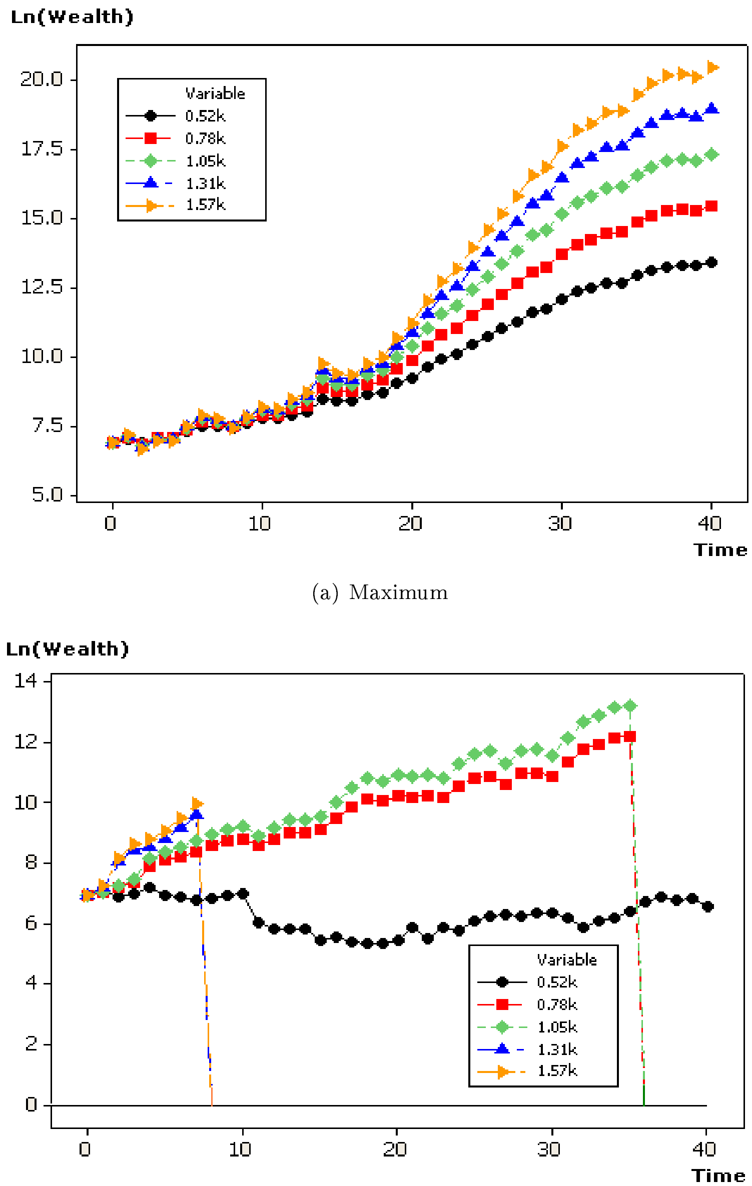
Table 7: Kelly Fractions for Bicksler-Thorp (1973) Case II

The results from the simulations with experiment 3 are contained in Table 8 and Figures 9, 10, and 11. This experiment is based on actual market returns. The striking aspects of the statistics in Table 8 are the sizable gains and losses. For the the most aggressive strategy ($1.57k$), it is possible to lose 10,000 times the initial wealth. This assumes that the shortselling is permissable through to the horizon.

Table 8: Final Wealth Statistics by Kelly Fraction for Bicksler-Thorp (1973) Case II

			Fraction			
Statistic	0.26k	0.52k	0.78k	1.05k	1.31k	1.57k
Max	65842.09	673058.45	5283234.28	33314627.67	174061071.4	769753090
Mean	12110.34	30937.03	76573.69	182645.07	416382.80	895952.14
Min	2367.92	701.28	-4969.78	-133456.35	-6862762.81	-102513723.8
St. Dev.	6147.30	35980.17	174683.09	815091.13	3634459.82	15004915.61
Skewness	1.54	4.88	13.01	25.92	38.22	45.45
Kurtosis	4.90	51.85	305.66	950.96	1755.18	2303.38
$> 5 \times 10$	3000	3000	2998	2970	2713	2184
10^2	3000	3000	2998	2955	2671	2129
$> 5 \times 10^2$	3000	3000	2986	2866	2520	1960
$> 10^3$	3000	2996	2954	2779	2409	1875
$> 10^4$	1698	2276	2273	2112	1794	1375
$> 10^5$	0	132	575	838	877	751
$> 10^6$	0	0	9	116	216	270

The highest and lowest final wealth trajectories are presented in Figures 9. In the worst case, the trajectory is terminated to indicate the timing of vanishing wealth. There is quick bankruptcy for the aggressive strategies.

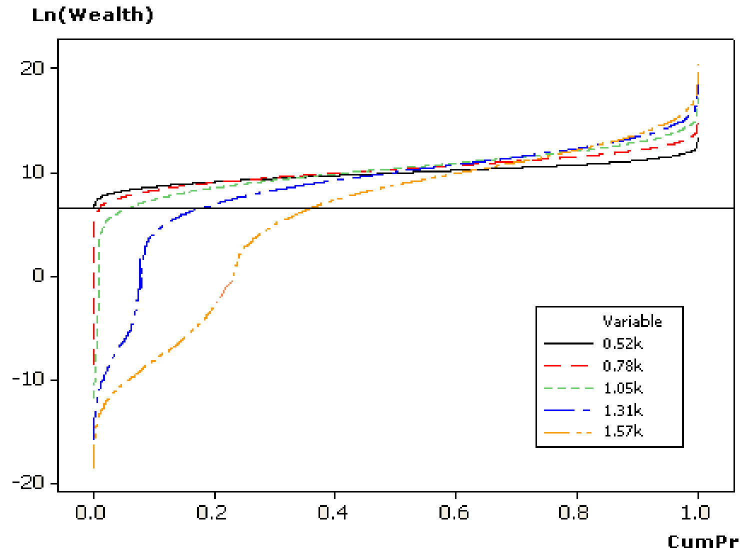


(a) Maximum

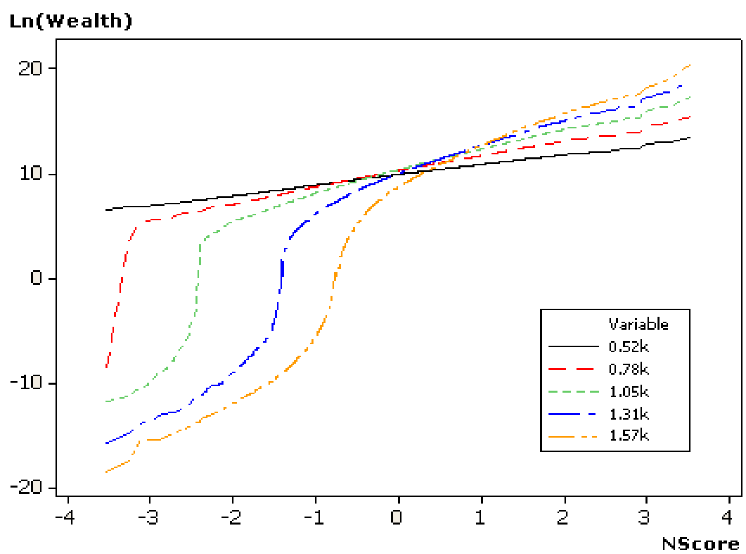
(b) Minimum

Figure 9: Trajectories with Final Wealth Extremes: Bicksler-Thorp (1973) Case II

The strong downside is further illustrated in the distribution of final wealth plot in Figure 10. The normal probability plots are almost linear on the upside (log-normality), but the downside is much more extreme than log-normal for all strategies except for $0.52k$. Even the full Kelly is risky in this case. The inverse cumulative distribution shows a high probability of large losses with the most aggressive strategies. In constructing these plots the negative growth was incorporated with the formula $growth = [signW_T]\, ln(|W_T|)$.



(a) Inverse Cumulative



(b) Normal Plot

Figure 10: Final Ln(Wealth) Distributions: Bicksler-Thorp (1973) Case II

The mean-standard deviation trade-off in Figure 11 provides more evidence to the riskyness of the high proportion strategies. When the fraction exceeds the full Kelly, the drop-off in growth rate is sharp, and that is matched by a sharp increase in standard deviation.

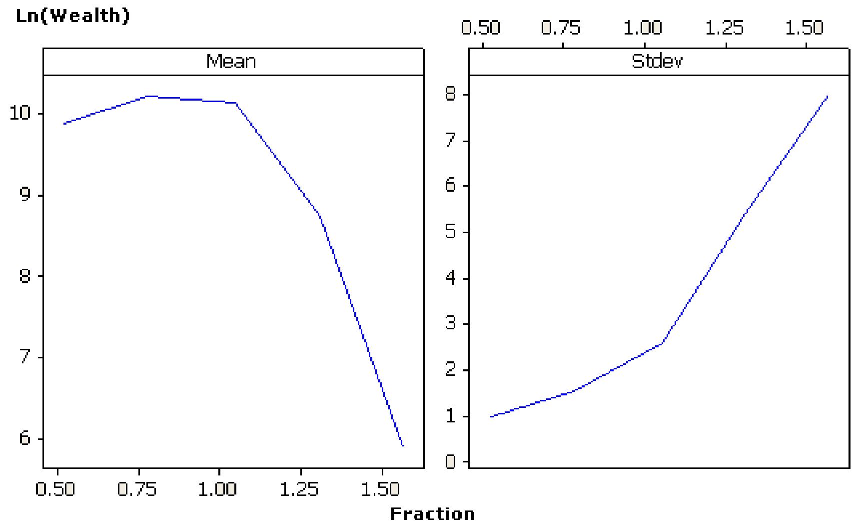


Figure 11: Mean-Std Tradeoff: Bicksler-Thorp (1973) Case II

The results in experiment 3 lead to the following conclusions.

1. The statistics describing the end of the horizon ($T = 40$) wealth are again monotone in the fraction of wealth invested in the Kelly portfolio. Specifically (i) the maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction; and (ii) the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. The growth and decay are pronounced and it is possible to have large losses. The fraction of the Kelly optimal growth strategy exceeds 1 in the most levered strategies and this is very risky. There is a trade-off between return and risk, but the mean for the levered strategies is growing far less than the standard deviation. The disadvantage of leveraged investment is clearly illustrated with the cumulative distribution in Figure 10. The log-normality of final wealth does not hold for the levered strategies.

2. The maximum and minimum final wealth trajectories clearly show the return - risk of levered strategies. The worst and best scenarios are the not same for all Kelly fractions. The worst scenario for the most levered strategy shows the rapid decline in wealth. The mean-standard deviation trade-off confirms the riskyness/folly of the aggressive strategies.

4 Discussion

The Kelly optimal capital growth investment strategy is an attractive approach to wealth creation. In addition to maximizing the rate of growth of capital, it avoids bankruptcy and overwhelms any essentially different investment strategy in the long run (MacLean, Thorp and Ziemba, 2009). However, automatic use of the Kelly strategy in any investment situation is risky. It requires some adaptation to the investment environment: rates of return, volatilities, correlation of alternative assets, estimation error, risk aversion preferences, and planning horizon. The experiments in this paper represent some of the diversity in the investment environment. By considering the Kelly and its variants we get a concrete look at the plusses and minusses of the capital growth model. The main points from the Bicksler and Thorp (1973) and Ziemba and Hausch (1986) studies are confirmed.

- The wealth accumulated from the full Kelly strategy does not stochastically dominate fractional Kelly wealth. The downside is often much more favorable with a fraction less than one.
- There is a tradeoff of risk and return with the fraction invested in the Kelly portfolio. In cases of large uncertainty, either from intrinsic volatility or estimation error, security is gained by reducing the Kelly investment fraction.
- The full Kelly strategy can be highly levered. While the use of borrowing can be effective in generating large returns on investment, increased leveraging beyond the full Kelly is not warranted. The returns from over-levered investment are offset by a growing probability of bankruptcy.
- The Kelly strategy is not merely a long term approach. Proper use in the short and medium run can achieve wealth goals while protecting against drawdowns.

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5 Appendix

The proportional investment strategies in the experiments of Bicksler and Thorp (1973) have fractional Kelly equivalents. The Kelly investment proportion for the experiments are deveolped in this appendix.

5.1 Kelly Strategy with Uniform Returns

Consider the problem

$$Max_x \{E(ln(1+r+x(R-r)\},$$

where R is uniform on $[a,b]$ and r =the risk free rate.

We have the first order condition

$$\int_a^b \frac{R-r}{1+r+x(R-r)} \times \frac{1}{b-a} dR = 0,$$

which reduces to

$$x(b-a) = (1+r)ln\left(\frac{1+r+x(b-r)}{1+r+x(a-r)}\right) \Longleftrightarrow \left[\frac{1+r+x(b-r)}{1+r+x(a-r)}\right]^{\frac{1}{x}} = e^{\frac{b-a}{1+r}}.$$

In the case considered in Experiment II, $a = -0.25, b = 0.5, r = 0$. The equation becomes $\left[\frac{1+0.5x}{1-0.25x}\right]^{\frac{1}{x}} = e^{0.75}$, with a solution $x = 2.8655$. So the Kelly strategy is to invest 286.55% of wealth in the risky asset.

5.2 Kelly Strategy with Normal Returns

Consider the problem

$$Max_x \{E(ln(1+r+x(R-r)\},$$

where R is Gaussian with mean μ_R and standard deviation σ_R, and r =the risk free rate. The solution is given by Merton (1990) as

$$x = \frac{\mu_R - r}{\sigma_R}.$$

The values in Experiment III are $\mu_R = 0.102, \sigma_R = 0.203, r = 0.039$, so the Kelly strategy is $x = 1.5288$.